Numerical Modeling of Earthquake Motions: Waves and Ruptures

June 23 - 27, 2024
Smolenice Castle, Slovakia

The finite-difference modeling: where is it and what's next?

Peter Moczo Jozef Kristek Jaroslav Valovcan
Martin Galis Miriam Kristekova

Comenius University Bratislava, Slovakia
and
Slovak Academy of Sciences, Slovakia

Introduction

if a function depends on time and space,
and is
discretized in time and space,
we must analyze
consequences of discretization
in
the **time**, **space**, **frequency** and **wavenumber** domains

Introduction

for decades,
FD schemes have been developed
without considering
the **wavenumber** domain

though the relevant analysis was, in fact,
possible already in 60s

if it was performed,
FD schemes today
could be
better (?)

Outline

homogeneous medium

- exact wavefield 1D
- spatially discretized wavefield
- consequence for spatial sampling
- governing equations in the WN domain
- FD scheme

heterogeneous medium

- exact wavefield 1D
- spatially discretized wavefield
- spatial discretization of a medium
- exact wavefield in a WN-limited medium
- consequence for FD modeling

homogeneous medium

ρ and C

are spatially independent

⬇

their wavenumber spectra are nonzero only at zero wavenumber

or, in other words,

their wavenumber spectra
are bounded by

$$k_\rho = k_C = 0$$

$$\rho \frac{\partial v(z,t)}{\partial t} = \frac{\partial \sigma(z,t)}{\partial z} \qquad C \frac{\partial \sigma(z,t)}{\partial t} = \frac{\partial v(z,t)}{\partial z}$$

assume a harmonic wave and transform to WN domain:

$$\omega \rho \tilde{V}(k,\omega) = -k \tilde{\Sigma}(k,\omega) \qquad \omega C \tilde{\Sigma}(k,\omega) = -k \tilde{V}(k,\omega)$$

⬇

wavefield wavenumber spectra

$$\tilde{V}(k,\omega) \qquad \tilde{\Sigma}(k,\omega)$$

are fully determined by a wavefield source

⬇

the wavefield is wavenumber limited

if the source spectrum is limited by some frequency wavenumber spectrum

of a function discretized with a grid spacing

h

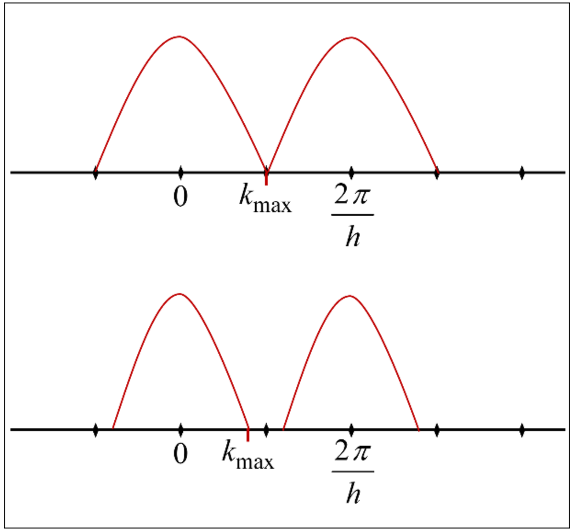
is periodic with period

$$2\frac{\pi}{h} =: 2\,k_N$$

to avoid aliasing

$$k_{\max} \leq \frac{\pi}{h} \equiv k_N$$

a discrete grid with a grid spacing
h
cannot support wavenumbers larger than
k_N



source and medium $\Rightarrow$ $k_{\max} = \frac{\omega_{\max}}{c}$ $\lambda_{\min} = \frac{2\pi}{k_{\max}}$

assuming grid spacing h

$$k_{\max} \leq k_N$$

$$\frac{2\pi}{\lambda_{\min}} \leq \frac{\pi}{h}$$

$$h \leq \frac{\lambda_{\min}}{2}$$

$$2 \leq \frac{\lambda_{\min}}{h} =: N_{\lambda \min}$$

at least 2 grid spacings
per minimum wavelength

because we can choose a source with spectrum limited to

$\omega_{\max}$

we may restrict to

$\mathcal{F}_{z \to k}$

$$\rho \frac{\partial \tilde{v}(k,t)}{\partial t} = -ik\,\tilde{\sigma}(k,t)$$

apply k_N – limited $\mathcal{F}^{-1}_{z \to k}$

⬇

$$\rho \frac{\partial v(z,t)}{\partial t} = \left[\frac{1}{hz}\cos\left(\frac{\pi z}{h}\right) - \frac{1}{\pi z^2}\sin\left(\frac{\pi z}{h}\right) \right] \overset{z}{*} \sigma(z,t)$$

spatial **derivatives** of $\sigma(z,t)$ and $v(z,t)$

are **replaced by** continuous, spatially unbounded **convolutions**

in a finite spatial grid, the convolutions must be approximated by **discrete convolutions**

this and a discrete approximation of the temporal derivative leads to **FD scheme**

analysis of stability and grid dispersion
yields
grid spacing and time step

!

in the homogeneous medium
FD-simulated wavefield
can be, depending on the FD scheme,
accurate up to
k_N

[*though, no so far developed FD scheme allows*
as few as 2 grid spacings
per minimum wavelength]

Outline

homogeneous medium

- exact wavefield 1D
- spatially discretized wavefield
- consequence for spatial sampling
- governing equations in the WN domain
- FD scheme

heterogeneous medium

- exact wavefield 1D
- spatially discretized wavefield
- spatial discretization of a medium
- exact wavefield in a WN-limited medium
- consequence for FD modeling

heterogeneous medium exact wavefield 1D

$$\rho(z)\frac{\partial v(z,t)}{\partial t} = \frac{\partial \sigma(z,t)}{\partial z} \qquad C(z)\frac{\partial \sigma(z,t)}{\partial t} = \frac{\partial v(z,t)}{\partial z}$$

assume a harmonic wave and transform to WN domain:

$$\omega\,\tilde{\rho}(k) * \tilde{V}(k,\omega) = -k\,\tilde{\Sigma}(k,\omega) \qquad \omega\,\tilde{C}(k) * \tilde{\Sigma}(k,\omega) = -k\tilde{V}(k,\omega)$$

assume that wavenumber spectra

$$\tilde{\rho}(k) \quad \tilde{C}(k) \qquad \tilde{V}(k) \quad \tilde{\Sigma}(k)$$

are bounded by wavenumbers

$$k_\rho \quad k_C \qquad k_V \quad k_\Sigma$$

then

$$k_\rho + k_V = k_\Sigma \qquad k_C + k_\Sigma = k_V$$

possibilities
in a heterogeneous medium

$$k_\rho = \infty \quad \text{and} \quad k_C = \infty$$

$$0 < k_\rho < \infty \quad \text{and/or} \quad 0 < k_C < \infty$$

$$\Rightarrow \quad k_V = k_\Sigma = \infty$$

in the heterogeneous medium,
wavenumber spectrum
of an exact wavefield
is **unbounded**

because

wavenumber spectrum
of an exact wavefield
is **unbounded**
in the heterogeneous medium

and

discrete grid with a grid spacing
h
cannot support
wavenumbers larger than
k_N

it is **in principle impossible**
to **simulate an exact wavefield**
in a **heterogeneous medium**
using the FD method

spatial discretization itself of a function

does not remove a potential wavenumber content

above

k_N

↓

medium must be

k_N – limited

before it is spatially discretized by a grid !

so,

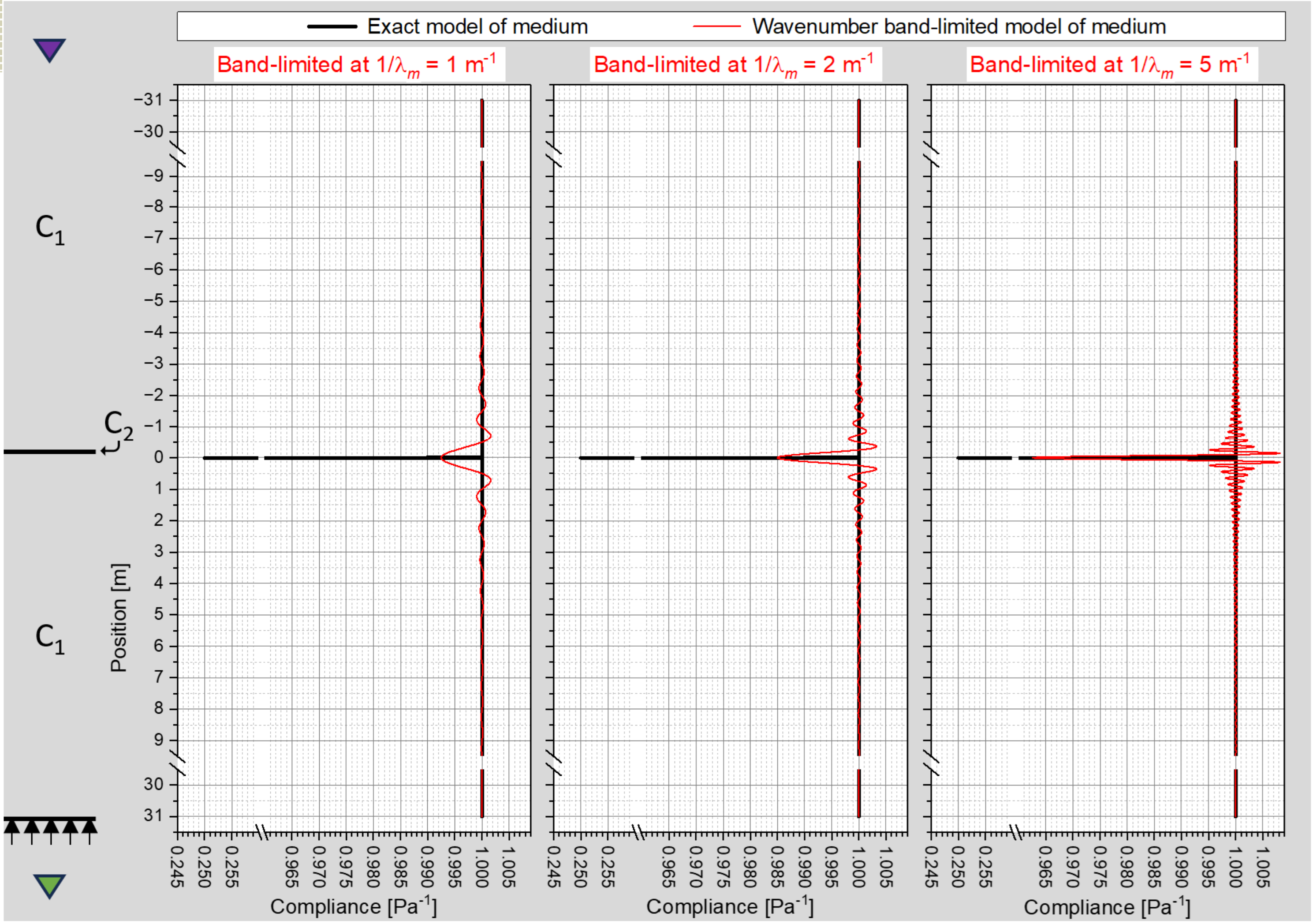
we must look at a wavefield

in a k_N – limited medium

heterogeneous medium

exact wavefield !

Model of Thin Layer Between Two Halfspaces



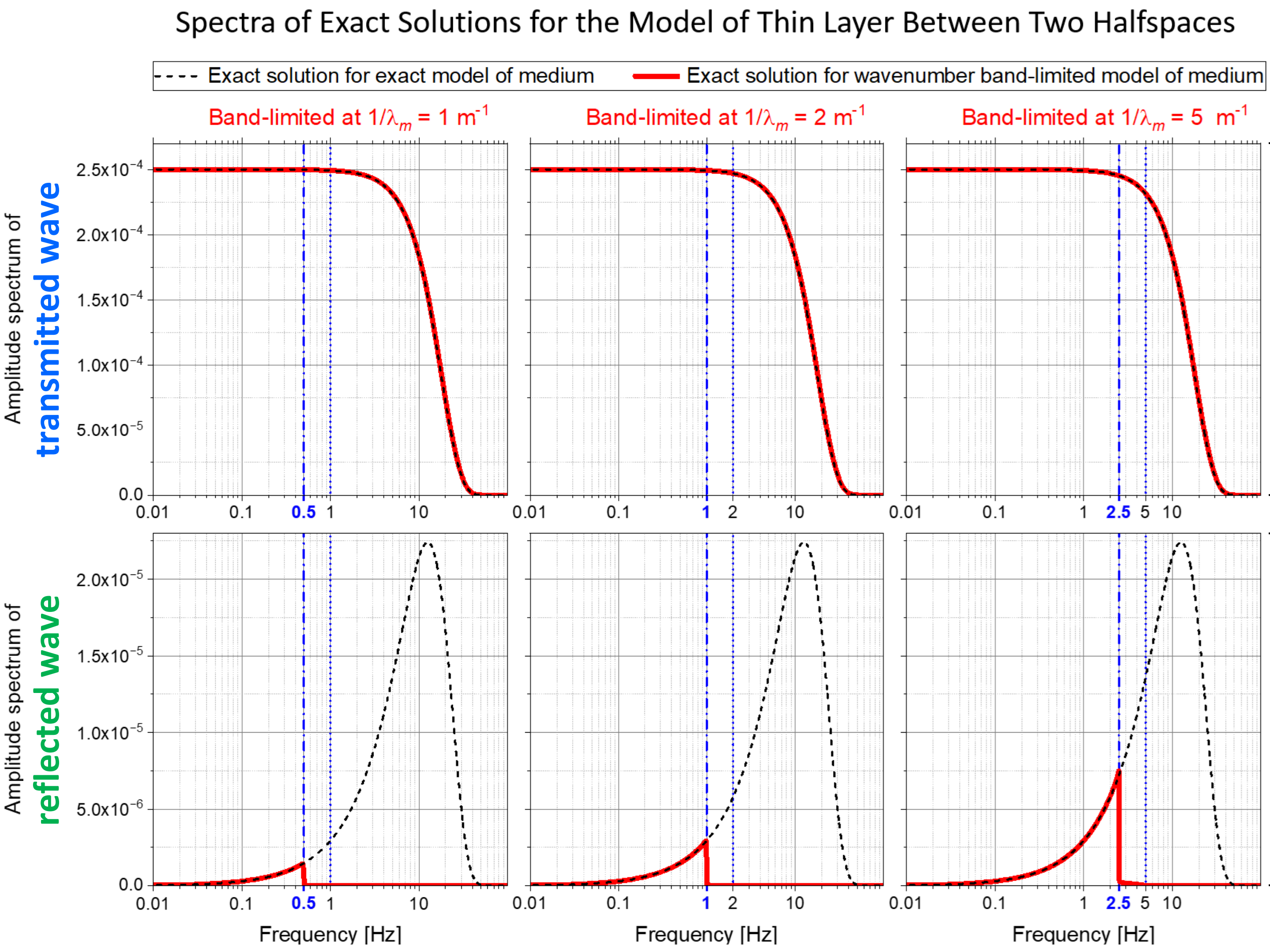
Exact model

$$C(z) = C_1 + (C_2 - C_1)\,\delta(z/\ell)$$

Wavenumber k_m-limited model

$$C_{k_m}(z) = C_1 + \frac{1}{\pi}\,\ell\,(C_2 - C_1)\ \textbf{Sinc}(k_m z)$$

heterogeneous medium exact wavefield !



Exact solutions for the k_m -limited model of medium

transmitted wave

the same as the transmitted wave
in the exact model
in the entire frequency range,
i.e.
in the entire corresponding range
of wavenumbers

reflected wave

exists and it is the same
as the reflected wave
in the exact model
only up to frequency $f_m/2$
i.e.
only up to wavenumber $k_m/2$

IMPLICATION of the finding on the exact wavefield

if we need the reflected wave
to exist and be accurate up to frequency

f_m

i.e. up to wavenumber

k_m

we must limit the model of medium
by at least wavenumber

$2k_m$

we still talk about the exact wavefield !

in the original model of medium and its k_m-limited version

is there
a consequence for the FD modeling ?

yes, there is !

let us recall <u>first</u> :

having an **exact** model of medium with a minimum speed

$$c_{\min}$$

we need to numerically simulate a wavefield
up to some maximum frequency

$$f_{\max}$$

this means that we have to sufficiently accurately simulate the wavefield for

$$f \le f_{\max} \qquad \lambda \ge \lambda_{\min} \qquad k \le k_{\max} = \frac{2\pi}{\lambda_{\min}}$$

as we know, we need a spatial grid spacing

$$h$$

such that

$$N_{\lambda_{\min}} = \frac{\lambda_{\min}}{h} \ge 2 \quad \text{and} \quad k_{\max} \le k_N = \frac{\pi}{h}$$

let us recall as a <u>second</u> :

because the grid with a spatial grid spacing

$$h$$

cannot support

wavenumbers larger than

$$k_N$$

we must limit our model of medium

in the WN domain by

$$k_N$$

and here comes
what we have found in the numerical experiment
with the exact wavefields :

if we want the wavefield
to be accurate up to wavenumber

$$k_N$$

we must limit the model of medium
by wavenumber

$$2k_N$$

if, however, spatial grid spacing

$$h$$

is needed for fulfilling condition

$$k \le k_N$$

then it is obvious that

twice smaller grid spacing

$$h/2$$

is needed for fulfilling condition

$$k \le 2k_N$$

Thus, we have

$$N_{\lambda_{\min}} = \frac{\lambda_{\min}}{\frac{h}{2}} = \frac{\lambda_{\min}}{h} 2 \ge \mathbf{4}$$

MAIN CONSEQUENCE AND CONCLUSION

no matter which FD scheme is used,
the simulation cannot be sufficiently accurate

if the medium is not $2k_N$ – limited
and minimum wavelength
is sampled by less than **4 spatial grid spacings**

what's next ?

a proper discrete representation
of a heterogeneous medium
and
the corresponding FD scheme

Thank you
for your attention

and, please, look at these articles,

there is much to learn 😉

Moczo, Kristek, Kristekova, Valovcan, Galis, Gregor (2022)

Material Interface in the Finite-Difference Modeling: A Fundamental View

Bull. Seismol. Soc. Am.

Valovcan, Moczo, Kristek, Galis, Kristekova (2023)

Can higher-order finite-difference operators
be applied across a material interface?

Bull. Seismol. Soc. Am.

Valovcan, Moczo, Kristek, Galis, Kristekova (2024)

How accurate numerical simulation of seismic waves
in a heterogeneous medium can be?

Bull. Seismol. Soc. Am.