New advances in implementing material heterogeneity in the finite-difference modelling

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Outline

- FDM
  - FDM – what is it ?
  - FDM among other numerical methods
  - Key conditions for competitiveness of FDM
- The problem: a discrete representation of heterogeneity of a medium in a heterogeneous FD scheme
  - Physics and mathematics of a discrete representation
  - Short excursion into a strange (though not unusual) and illuminating history
- The recent most advanced approaches
- Conclusions

FDM

as applied to seismic wave propagation

and earthquake ground motion

= a large diverse family

of computational schemes

based on

FD approximations

of the equation of motion and constitutive law

at space-time grid points

a state-of-the-art FD scheme

can significantly differ from some other FD scheme

in accuracy and computational efficiency

in a strongly heterogeneous medium

FDM, SEM and DGM

are the most important recent numerical-modelling methods

FDM is clearly dominant

in the seismic prospecting

It could be also dominant

(due to accuracy_and_efficiency)

in investigation of earthquake ground motion

in local surface sedimentary structures

if

all the schemes being used

were at the state-of-the-art level

Key conditions for competitiveness of FDM

an explicit heterogeneous FD scheme

on a uniform spatial grid

(the latter does not contradict the use
of an efficient discontinuous grid
composed of several uniform grids)

efficiency_and_accuracy
are determined by the
grid dispersion
and
discrete grid representation
of a material heterogeneity, mainly material interfaces

explicit	the field variable at a space-time grid point is calculated using an explicit FD formula that uses only values of the field variables at previous time levels
heterogeneous	one scheme is used for all interior grid points no matter what their positions are with respect to material interfaces presence of interfaces is accounted for only by values of the effective material parameters assigned to grid positions
uniform-grid	for a chosen computational region and maximum frequency one grid can be used for arbitrary alterations of geometry/position of material interfaces

Obviously,

a heterogeneous FD scheme

should approximate

an equation of motion and constitutive law

valid and having the same form

at any point of a medium

--

away of an interface or at an interface

in other words,

an interface should be represented

by an averaged medium

consistent with boundary conditions on the interface

Have a quick look at history:

strange (though not unusual) but illuminating and instructive recall:

an interface should be represented
by an averaged medium
consistent with boundary conditions on the interface

Was this the basis for developing heterogeneous FD schemes ?

No, it was not.

Is this now the basis for developing heterogeneous FD schemes ?

Very rarely.

recall:

an interface should be represented
by an averaged medium
consistent with boundary conditions on the interface

Was this the basis for developing heterogeneous FD schemes ?
No, it was not.
Is this now the basis for developing heterogeneous FD schemes ?
Very rarely.

Why?
The explanation would be rather critical
and related to
persisting unawareness, overlooking and need to sell own schemes and results
(*sorry*)
All this is supported by arguments on uncertainties of different kinds.

The historical development
was strongly influenced by the fact that

since 60‘s until 1984-1988 (introduction of a staggered grid)
developers FD-approximated
the 2^{nd}-order equation of motion in displacement
which certainly was not an easy task

(the only reasonable help
came from Tikhonov and Samarskii
who used a mathematical trick
to avoid spatial differentiation of elastic moduli)

After introduction

of the 1st-order velocity-stress formulation on a staggered grid

(which removed the problem of the 2nd-spatial derivatives of elastic moduli)

most developers assumed (and explicitly wrote)

that

the problem of implementing any heterogeneity

is solved implicitly

–

which obviously is not true

In parallel,

there had been another historical development

since 60‘s …

Backus (1962) found out
how to replace a stack of finely layered medium
by an averaged medium
consistent with boundary conditions
at the interfaces between layers.

Schoenberg & Muir (1989) extended the Backus approach
to arbitrary anisotropic layers.

Neither Backus nor Schoenberg & Muir
mentioned a relation to FD modelling.

Explicit essential reference to FD modelling
was done by Muir et al. (1992).

Nevertheless,
until the article by
Moczo, Kristek, Vavryčuk, Archuleta and Halada (2002),
the concept of an averaged medium
consistent with the interface boundary conditions
did not impact
the heterogeneous FD schemes.

Even then
many modellers (keen to sell own schemes?)
have been somehow “overlooking”
the evident and necessary progress.

This concludes

the short history of two non-communicating developments ☺

Back to the present

We have developed

unified discrete representations

of a strong material heterogeneity for

elastic

viscoelastic

poro-viscoelastic

(low-frequency approximation)

poro-viscoviscoelastic

(full frequency range)

media

The representations have capability of

a sub-cell resolution

and thus allow for

an arbitrary shape and position of an interface in a grid

Our principles

of finding an averaged medium

consistent with boundary conditions on an interface

1

The stiffness matrix of the averaged medium

has to have the same structure

(the same number of nonzero elements)

as the stiffness matrix of a smooth medium has

(except that the number of independent elements may be different)

consequence:

the number of algebraic operations

for updating stress

is the same as for the smooth medium

Our principles

of finding an averaged medium

consistent with boundary condition on an interface

2

If a grid cell contains a planar interface
between two homogeneous materials
perpendicular to a coordinate axis,
the stiffness matrix of the averaged medium in the cell
will correspond to the exactly averaged medium
accounting correctly
for the position of the interface in the grid cell.

Example

2D P-SV problem

poroelastic medium (low-frequency approximation)

boundary conditions

continuity of the

traction vector

fluid pressure

solid displacement vector

normal component
of the relative fluid displacement vector

constitutive law

smooth

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\\ -p\end{bmatrix}=\begin{bmatrix}\Lambda+\alpha^2M & \lambda+\alpha^2M & 0 & \alpha M\\ \lambda+\alpha^2M & \Lambda+\alpha^2M & 0 & \alpha M\\ 0 & 0 & 2\mu & 0\\ \alpha M & \alpha M & 0 & M\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \varepsilon_{xz}\\ \varepsilon_{w}\end{bmatrix}$$

averaged

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\\ -p\end{bmatrix}=\begin{bmatrix}XX+\dfrac{XP\ XP}{\Psi} & XZ+\dfrac{XP\ ZP}{\Psi} & 0 & \dfrac{XP}{\Psi}\\ XZ+\dfrac{XP\ ZP}{\Psi} & ZZ+\dfrac{ZP\ ZP}{\Psi} & 0 & \dfrac{ZP}{\Psi}\\ 0 & 0 & 2\langle\mu\rangle^{Hx} & 0\\ \dfrac{XP}{\Psi} & \dfrac{ZP}{\Psi} & 0 & \dfrac{1}{\Psi}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \varepsilon_{xz}\\ \varepsilon_{w}\end{bmatrix}$$

where, for example, $XX=\left\langle\left\langle\Lambda-\dfrac{\lambda^2}{\Lambda}\right\rangle^{z}+\left(\left\langle\dfrac{\lambda}{\Lambda}\right\rangle^{z}\right)^{2}\left\langle\Lambda\right\rangle^{Hz}\right\rangle^{Hx}$ $\quad XZ=\left\langle\dfrac{\lambda}{\Lambda}\right\rangle^{xz}\left\langle\Lambda\right\rangle^{Hxz}$

equation of motion

smooth

$$\begin{bmatrix} \dot{v}_x \\ -\dot{q}_x \\ \dot{v}_z \\ -\dot{q}_z \end{bmatrix} = \frac{1}{\left(\frac{\rho}{\rho_f} - \frac{\rho_f}{m}\right)} \begin{bmatrix} \frac{1}{\rho} & \frac{1}{m} & \frac{b}{m} & 0 & 0 & 0 \\ \frac{1}{m} & \frac{\rho}{\rho_f}\frac{1}{m} & \frac{\rho}{\rho_f}\frac{b}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho} & \frac{1}{m} & \frac{b}{m} \\ 0 & 0 & 0 & \frac{1}{m} & \frac{\rho}{\rho_f}\frac{1}{m} & \frac{\rho}{\rho_f}\frac{b}{m} \end{bmatrix} \begin{bmatrix} \sigma_{xx},_x + \sigma_{xz},_z \\ p,_x \\ q_x \\ \sigma_{zz},_z + \sigma_{xz},_x \\ p,_z \\ q_z \end{bmatrix}$$

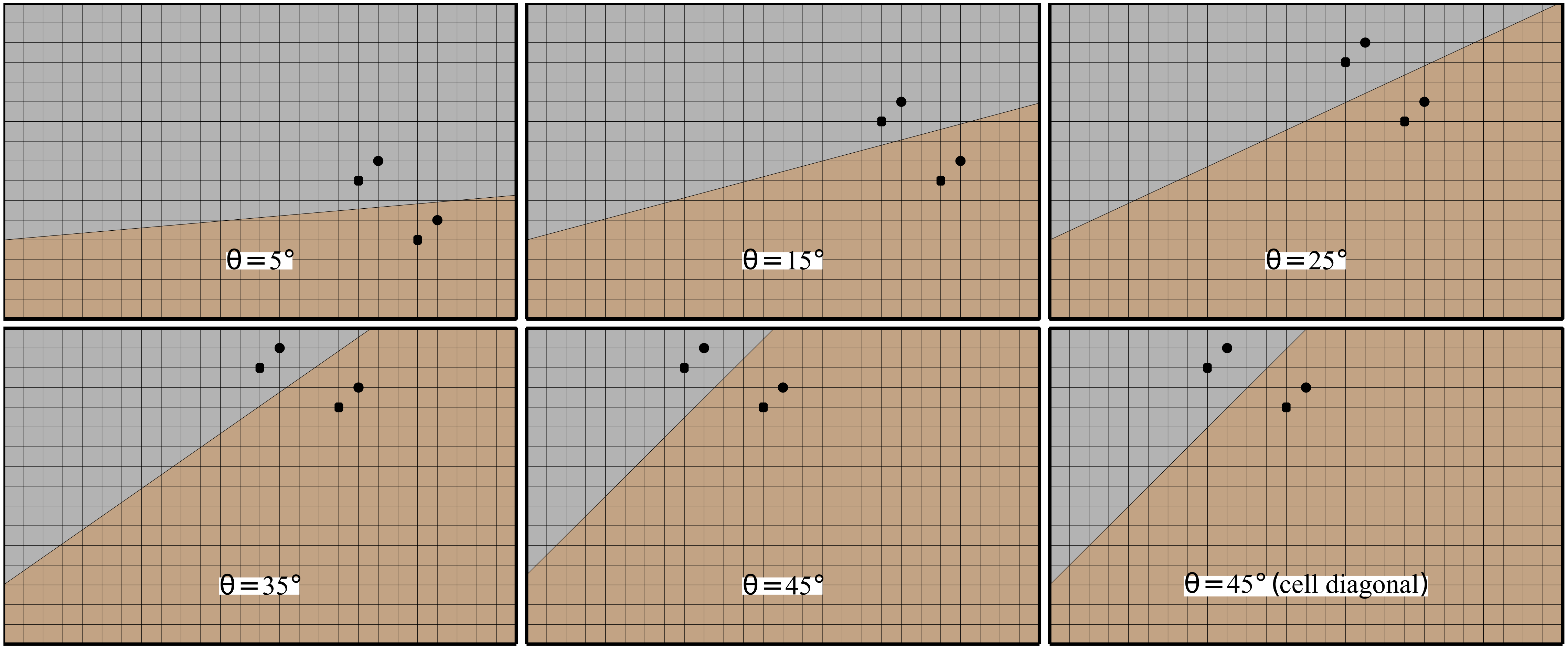
averaged

$$\begin{bmatrix} \dot{v}_x \\ -\dot{q}_x \\ \dot{v}_z \\ -\dot{q}_z \end{bmatrix} = \begin{bmatrix} \frac{\langle F^x\rangle^z}{\langle S^x\rangle^z} & \frac{\langle G^x\rangle^z}{\langle S^x\rangle^z} & \frac{\langle H^x\rangle^z}{\langle S^x\rangle^z} & 0 & 0 & 0 \\ \frac{\langle R^x\rangle^z \left\langle \frac{G^x}{R^x}\right\rangle^z}{\langle S^x\rangle^z} & \frac{\langle P^x\rangle^z \langle G^x\rangle^z}{\langle S^x\rangle^z} & \frac{\langle P^x\rangle^z \langle H^x\rangle^z}{\langle S^x\rangle^z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\langle F^z\rangle^x}{\langle S^z\rangle^x} & \frac{\langle G^z\rangle^x}{\langle S^z\rangle^x} & \frac{\langle H^z\rangle^x}{\langle S^z\rangle^x} \\ 0 & 0 & 0 & \frac{\langle R^z\rangle^x \left\langle \frac{G^z}{R^z}\right\rangle^x}{\langle S^z\rangle^x} & \frac{\langle P^z\rangle^x \langle G^z\rangle^x}{\langle S^z\rangle^x} & \frac{\langle P^z\rangle^x \langle H^z\rangle^x}{\langle S^z\rangle^x} \end{bmatrix} \begin{bmatrix} \sigma_{xx},_x + \sigma_{xz},_z \\ p,_x \\ q_x \\ \sigma_{zz},_z + \sigma_{xz},_x \\ p,_z \\ q_z \end{bmatrix}$$

where, for example,

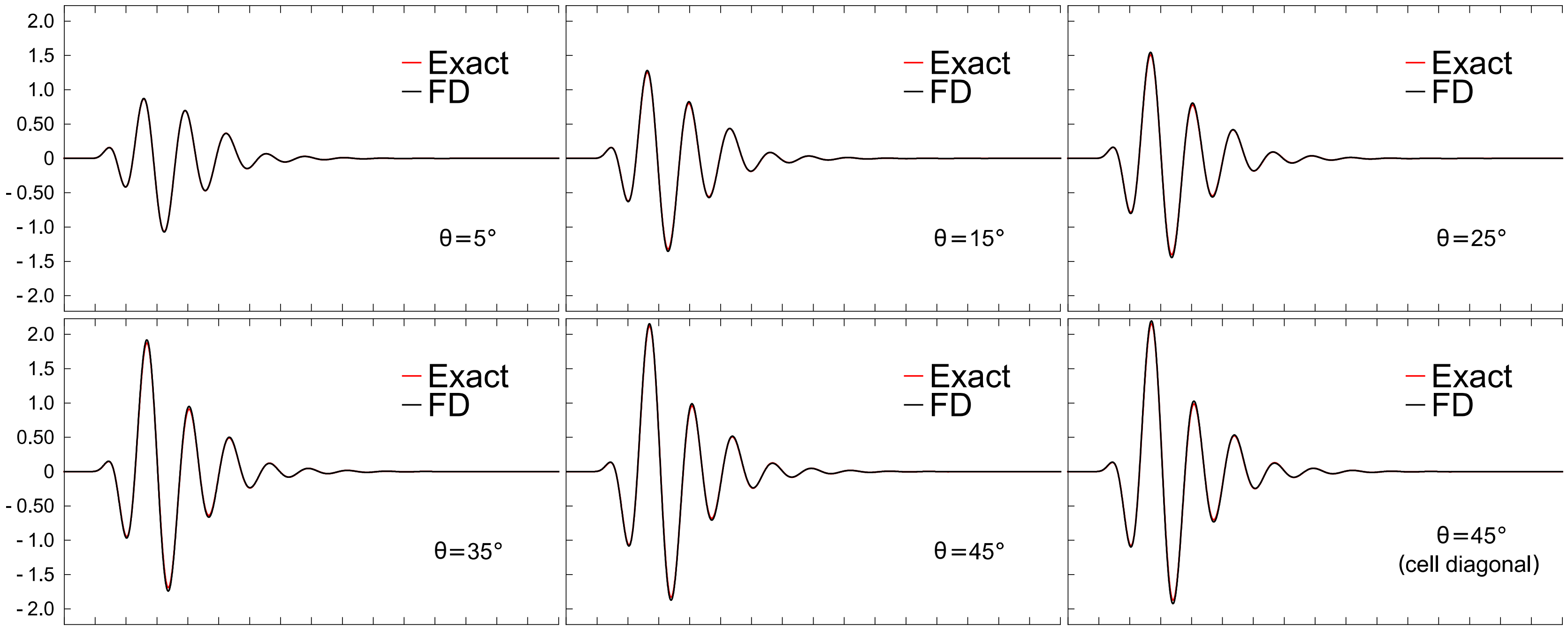
$$F^\xi = \frac{1}{\langle \rho_f \rangle^\xi} \qquad H^\xi = \frac{\langle b \rangle^\xi}{\langle m \rangle^\xi}$$

poroelastic halfspaces with an oblique planar interface



v_x

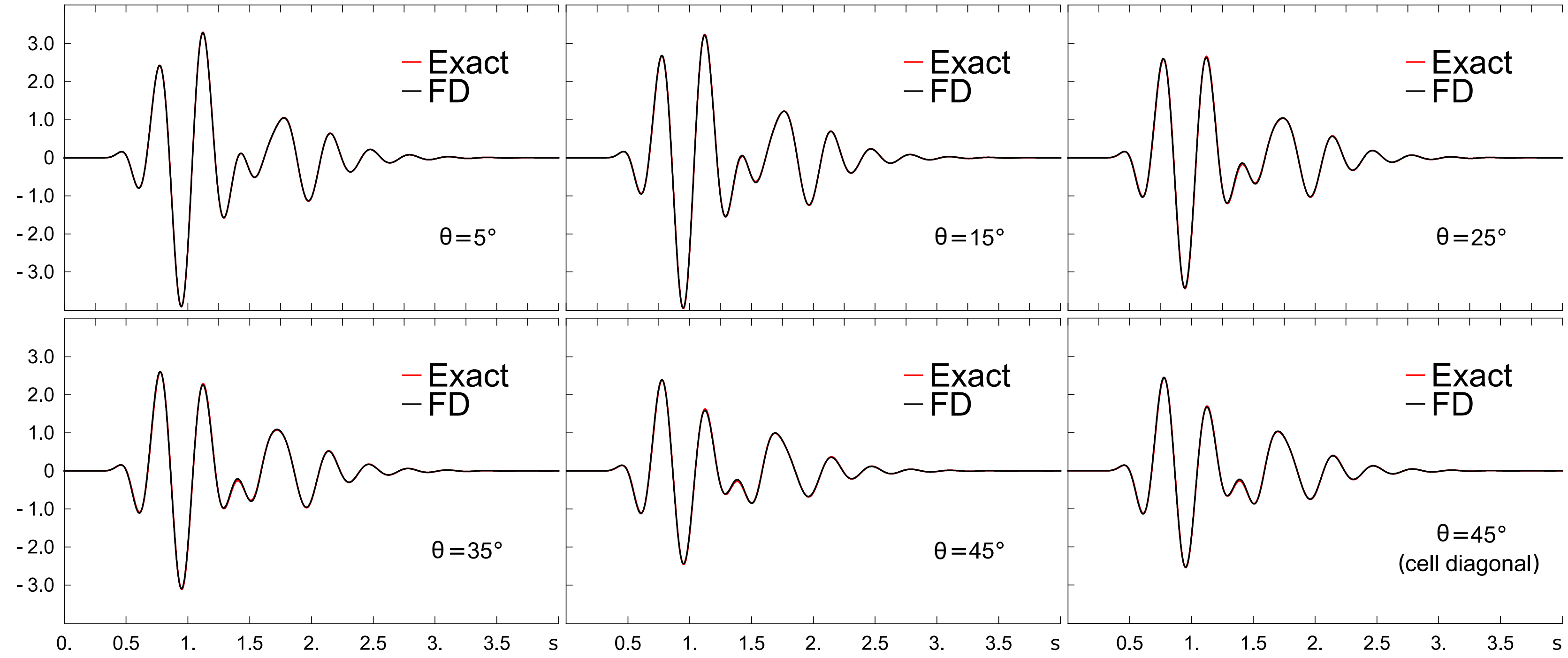
receiver in the upper halfspace



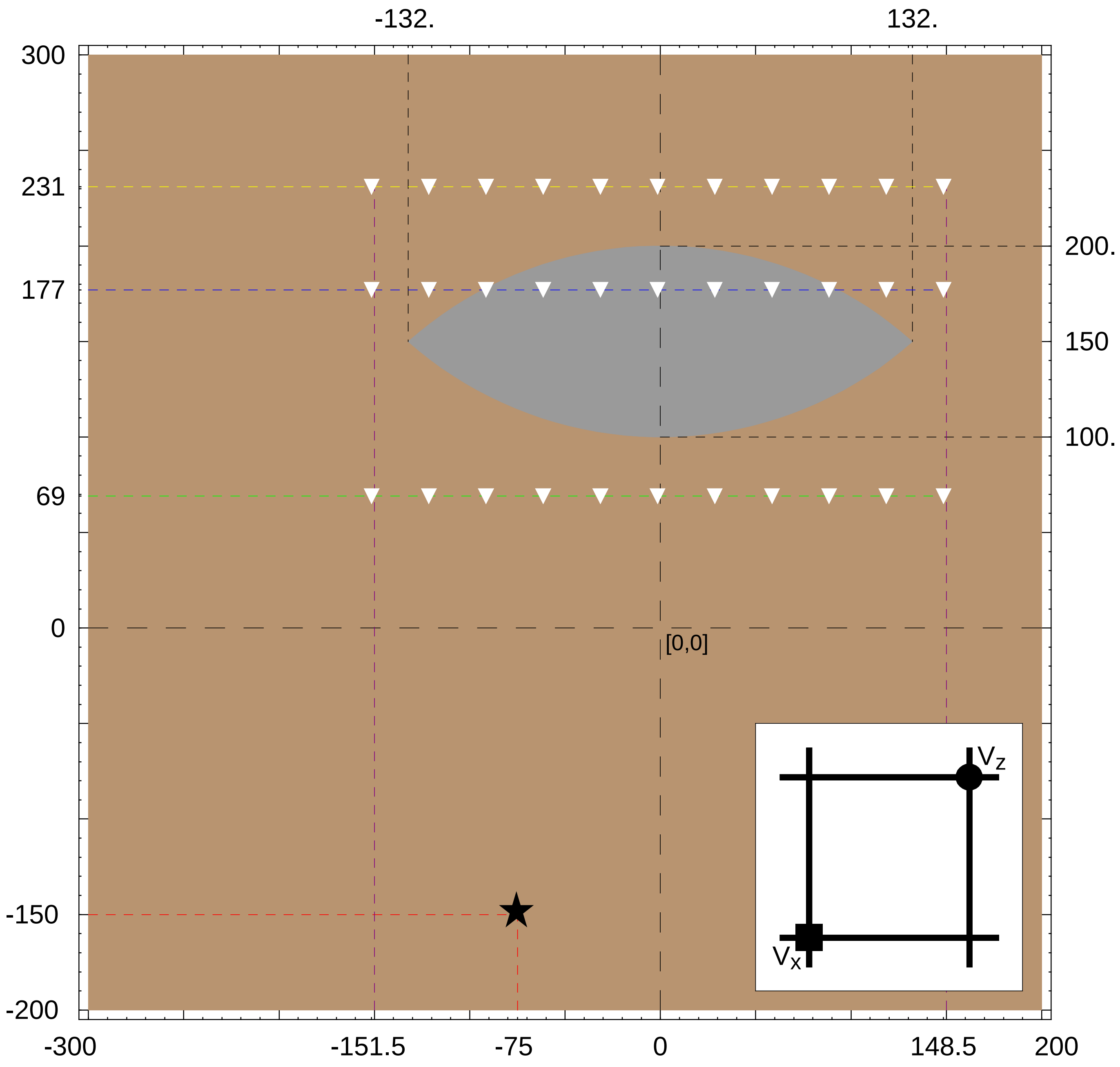
Exact:
using code Gar6more2D
(Diaz & Ezziani 2008)

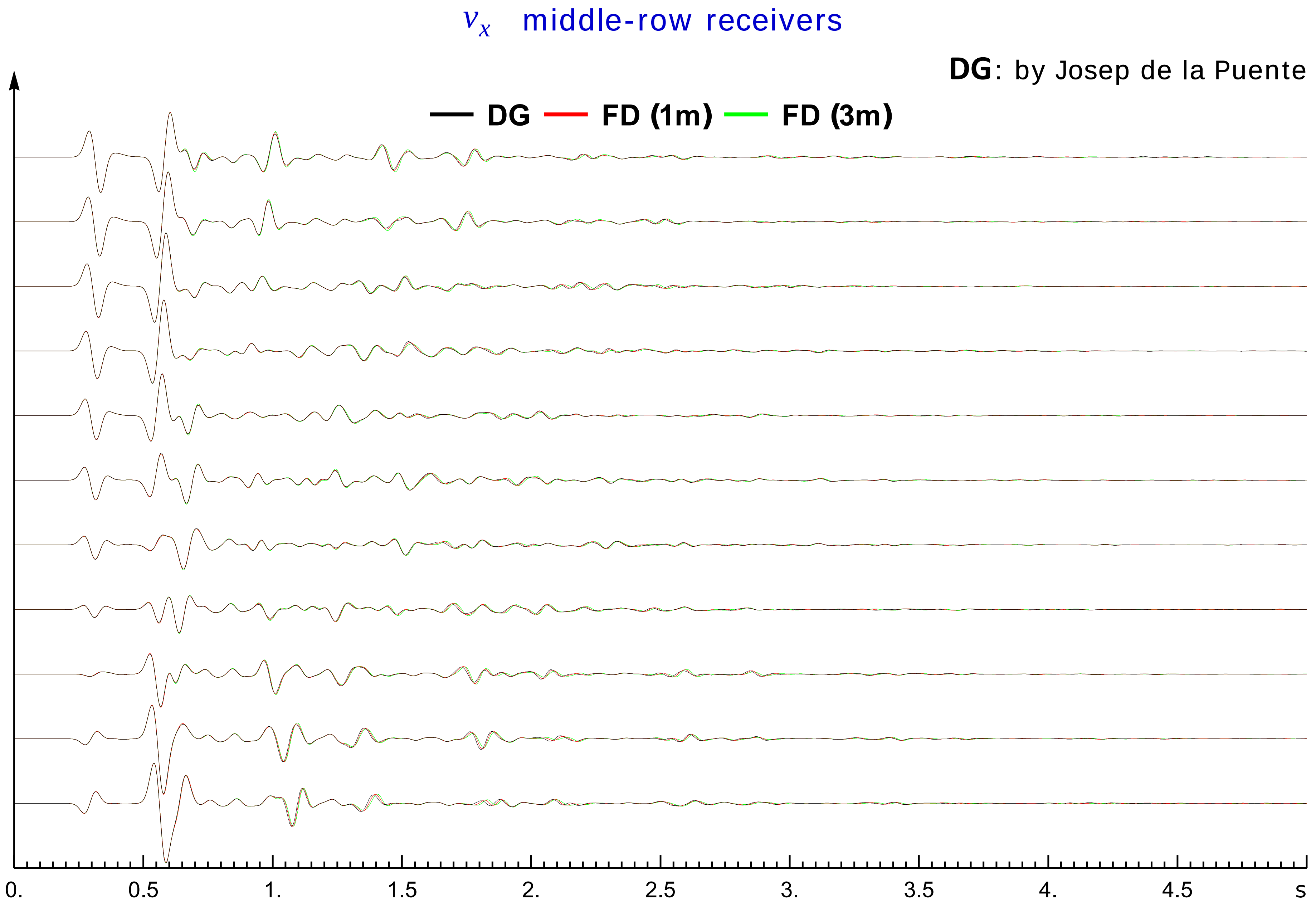
v_x

receiver in the lower halfspace



poroelastic lens
in a poroelastic halfspace





v_x middle-row receivers
DG: by Josep de la Puente
DG
FD (1m)
FD (3m)
0.
0.5
1.
1.5
2.
2.5
3.
3.5
4.
4.5
s

We have performed detailed comparisons

against SEM (Emmanuel Chaljub, Florent De Martin)

for

stringent **viscoelastic** models specially designed

for testing accuracy, sub-cell resolution and computational efficiency

of our discrete representation

The models included

sophisticated canonical configurations

and

complex models of the Mygdonian basin near Thessaloniki

The comparisons confirmed

accuracy, sub-cell resolution and computational efficiency

of our discrete representation

Because we have

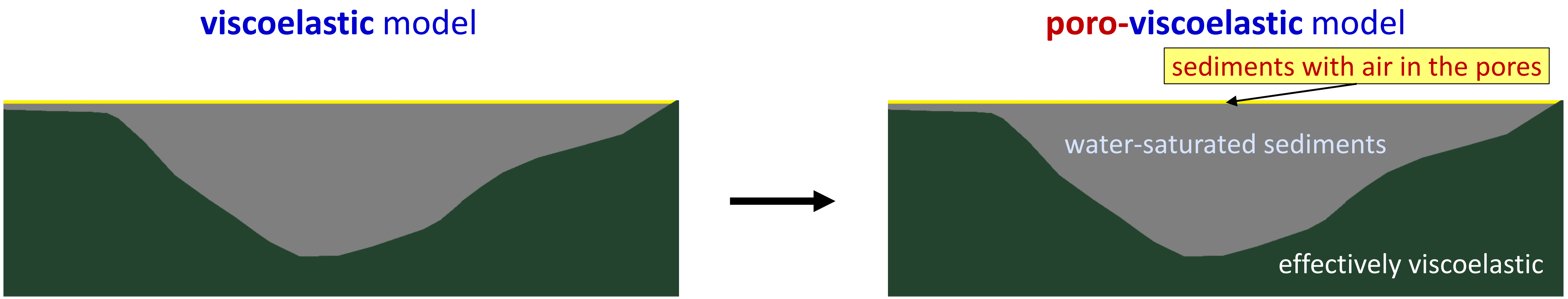
an efficient and sufficiently accurate

representation of heterogeneity of the poro-viscoelastic medium

we can apply this representation

also to

models with viscoelastic and poro-viscoelastic parts



this can be achieved by

- setting appropriate values of porosity and constant permeability
- choosing an appropriate fluid
- setting an appropriate value of the bulk modulus for a solid phase
- calculating solid-phase density from the density of bedrock
- calculating shear and bulk moduli
- setting an appropriate tortuosity

conclusions

with a sufficiently accurate discrete representation of a material heterogeneity

the most advanced FD schemes

are more efficient

for modelling earthquake ground motion

in local surface sedimentary structures

than the spectral-element and discontinuous-Galerkin methods

Thank you for your attention

Moczo, Kristek, Gális 2014 The Finite-difference Modelling of Earthquake Motions: Waves and Ruptures *Cambridge University Press*

Kristek, Moczo, Chaljub, Kristekova 2017 An orthorhombic representation of a heterogeneous medium for the finite-difference modelling of seismic wave propagation *Geophys. J. Int.* 208, 1250–126

Moczo, Gregor, Kristek, de la Puente 2019 A discrete representation of material heterogeneity for the finite-difference modelling of seismic wave propagation in a poroelastic medium *Geophys. J. Int.* 216, 1072-1099.

Kristek, Moczo, Chaljub, Kristekova 2019 A discrete representation of a heterogeneous viscoelastic medium for the finite-difference modelling of seismic wave propagation *Geophys. J. Int.,* in press.

EXAMPLE: 2D P-SV constitutive law and equations of motion

poroelastic medium (low-frequency approximation)

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \\ -p \end{bmatrix} = \begin{bmatrix} \Lambda+\alpha^2 M & \lambda+\alpha^2 M & 0 & \alpha M \\ \lambda+\alpha^2 M & \Lambda+\alpha^2 M & 0 & \alpha M \\ 0 & 0 & 2\mu & 0 \\ \alpha M & \alpha M & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \varepsilon_{xz} \\ \varepsilon_{w} \end{bmatrix}$$

$$\begin{aligned} \rho \dot{v}_x &= \sigma_{xx},_x + \sigma_{xz},_z - \rho_f \dot{q}_x \\ \rho \dot{v}_z &= \sigma_{xz},_x + \sigma_{zz},_z - \rho_f \dot{q}_z \\ m \dot{q}_x &= -p,_x - \rho_f \dot{v}_x - b q_x \\ m \dot{q}_z &= -p,_z - \rho_f \dot{v}_z - b q_z \end{aligned}$$

$\sigma_{xx}, \sigma_{zz}, \sigma_{xz}$	total stress-tensor components
p	fluid pressure
$\varepsilon_{xx}, \varepsilon_{zz}, \varepsilon_{xz}$	solid matrix strain-tensor components
w_x, w_z ; $\varepsilon_w \equiv w_x,_x + w_z,_z$	components of displacement of the fluid relative to the solid frame
$\Lambda \equiv \lambda + 2\mu$	Lamé elastic coefficients of the solid matrix
α	poroelastic coefficient of effective stress
M	coupling modulus between the solid and fluid
v_x, v_z	solid particle velocities
q_x, q_z	fluid particle velocities relative to the solid
ρ, ρ_f	total and fluid densities
m	mass coupling coefficient
b	resistive friction

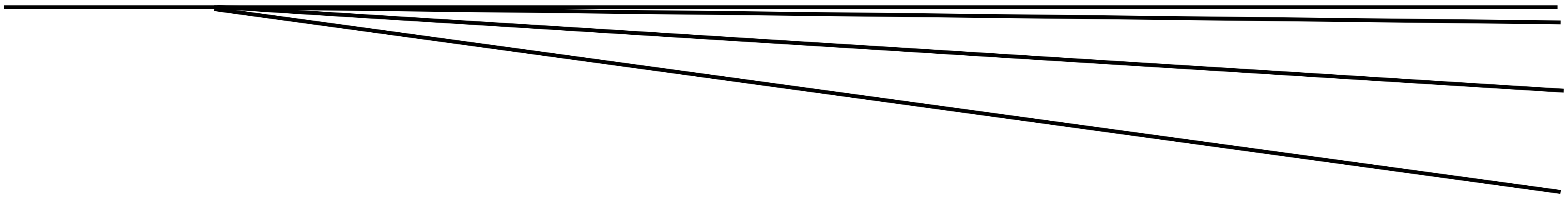
boundary conditions at an interface between

poroelastic media

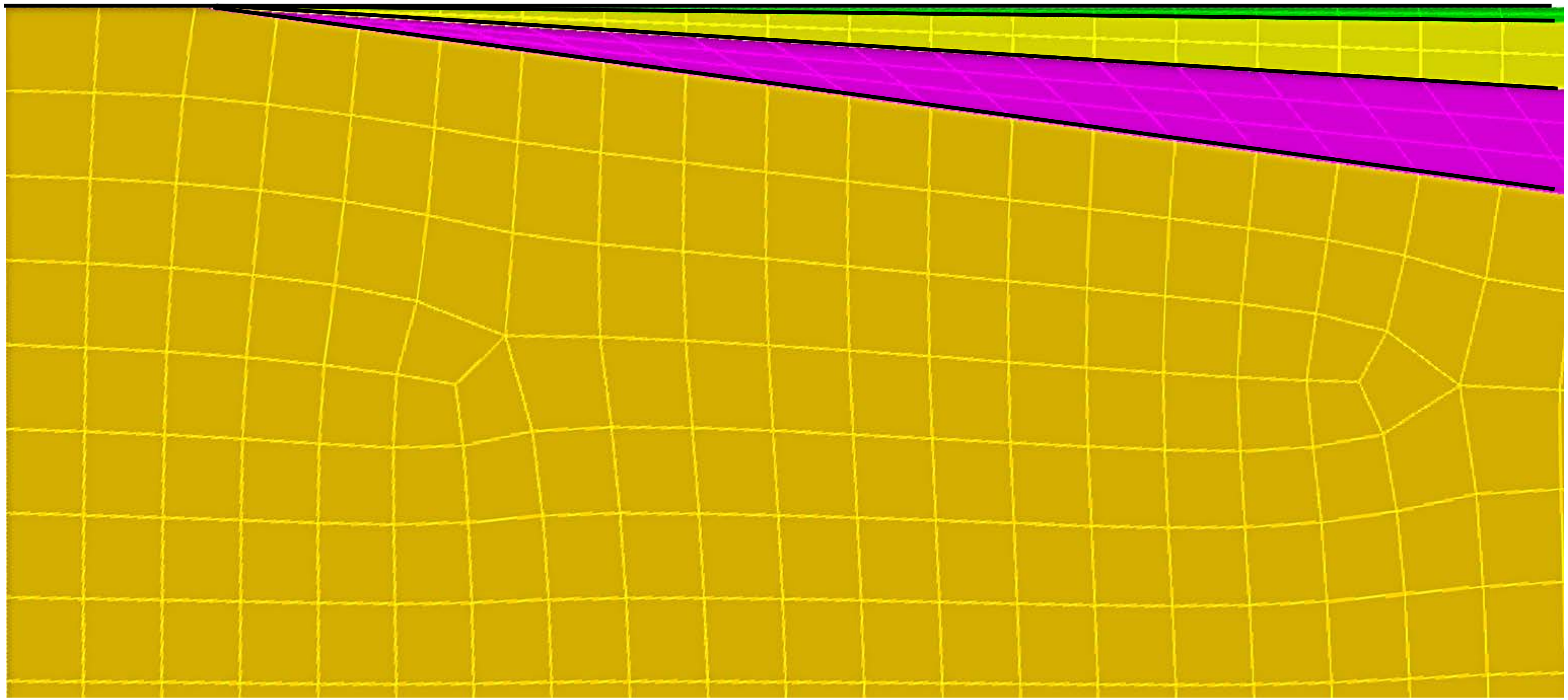
continuity of the

$\sigma_{ij}^{+} n_j = \sigma_{ij}^{-} n_j$	traction vector
$p^{+} = p^{-}$	fluid pressure
$u_i^{+} = u_i^{-}$	solid displacement vector
$w_i^{+} n_i = w_i^{-} n_i$	normal component of the relative fluid displacement vector

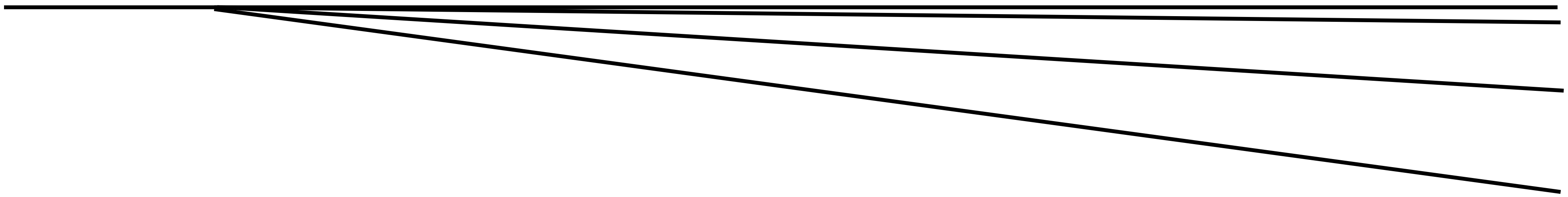
interfaces in a sedimentary wedge



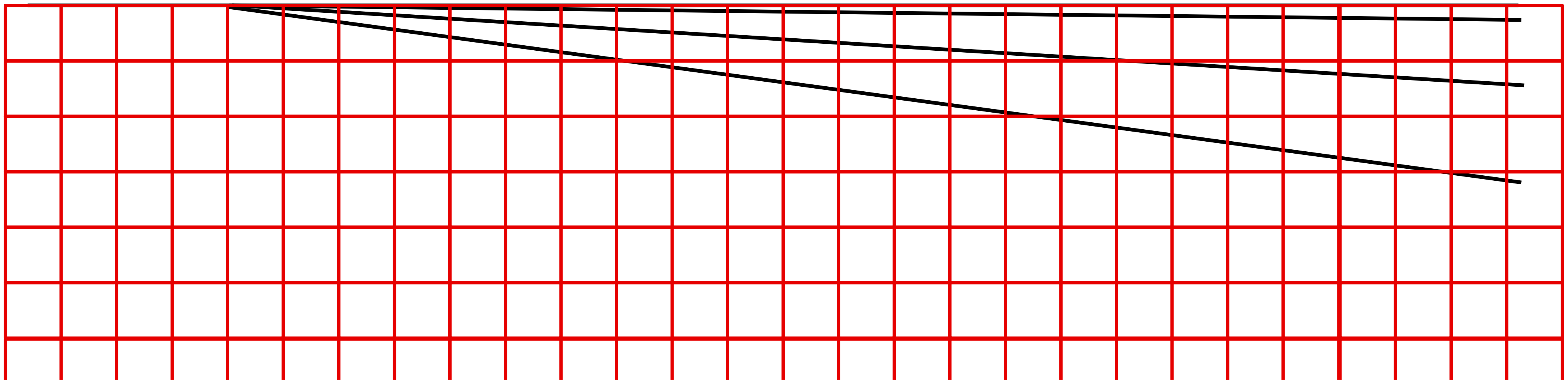
interfaces in a sedimentary wedge



Florent De Martin EFISPEC3D available at http://efispec.free.fr interfaces in a sedimentary wedge



interfaces in a sedimentary wedge



SPEM minimum node-to-node distance: 0.1 m

FDM grid spacing: 5.0 m