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EFFICIENCY AND OPTIMIZATION OF THE 3-D FINITE-DIFFERENCE MODELING OF SEISMIC GROUND MOTION*

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We present a tutorial introduction to the 3-D finite-difference modeling of seismic ground motion in elastic and viscoelastic media with special emphasis on its computational efficiency. We consider four basic types of the finite-difference schemes — the displacement-stress, displacement-velocity-stress and velocity-stress schemes on a staggered grid, and displacement scheme on a conventional grid. Their memory requirements in the case of perfectly elastic medium, elastic medium with aposteriori approximate attenuation correction, and realistic viscoelastic medium are reviewed. We also present application of the powerful optimization techniques to the 3-D fourth-order displacement-stress and displacement-velocity-stress modeling in the case of viscoelastic medium whose rheology is based on the generalized Maxwell body. Description of a medium using material cell types and use of a discontinuous grid with combined memory optimization makes it possible to simulate earthquake ground motion in realistic large-scale models.

1. Introduction

Three-dimensional (3-D) numerical modeling of seismic wave propagation and earthquake ground motion has become a necessary tool in a majority of seismological investigations and seismic exploration. There is a large variety of computational methods that differ one from another by range of applicability, accuracy, and efficiency. The existence of many computational methods indicates that, in fact, none of them is really universal — applicable to all medium-wavefield configurations with sufficient accuracy and efficiency.

In modeling earthquake ground motion it is often necessary to include a point dislocation source or realistic fault model, complex heterogeneity of the medium (that includes sharp interfaces, large velocity contrasts and high Poisson's ratio), and topography of the free surface. The finite-difference, finite-element, boundary-integral, discrete-wavenumber, spectral-element and boundary-element methods are good examples of different approaches to model earthquake ground motion. They can be divided into three groups — boundary,

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domain and hybrid methods. As Takenaka *et al.*[1] pointed out, the boundary methods are more accurate while domain methods are applicable to more complex (i.e., more realistic) models. In the 3-D case, boundary methods, due to their computational cost, are usually limited to one or two homogeneous layers. Combining two or more methods in the hybrid modeling allows overcoming limitations of the single method. For a relatively detailed review of many recent methods we refer to the article by Takenaka *et al.*[1] A concise overview of the methods can be found in the paper by Komatitsch and Tromp.[2]

Finite-difference method, one of the best-known domain methods, has been recognized for almost three decades as a powerful method for numerical simulation of seismic wave propagation and earthquake ground motion. The reason for it is the relative simplicity and robustness of the method. The finite-difference method is applicable to complex media and easy to implement in computer codes.

The importance of the finite-difference method has been recently proven by its role in three-dimensional modeling of earthquake ground motion in large sedimentary basins — e.g., see Refs. 3–7. Very recently, the finite-difference method played the key role in the Northridge and Kobe simultaneous simulation experiments (e.g., see Ref. 8).

There are two well-known drawbacks of the finite-difference method. One is the principal difficulty to implement boundary conditions on geometrically complex interfaces, especially the traction-free condition on the free-surface topography. Since this is an inherent problem of the finite-difference method, one reasonable way to overcome it is to combine the finite-difference method with the finite-element method.[9]

The second problem arises when the finite-difference method is applied to large-scale models as, e.g., Los Angeles or Osaka sedimentary basins. In such cases the method requires very large computer memory and time. This imposes serious limits on application of the finite-difference method.

Thus, it is obvious that memory optimization, sophisticated encoding and parallelization are necessary to facilitate further progress in the finite-difference modeling of earthquake ground motion.

The purpose of this article is to provide a tutorial introduction into the 3-D finite-difference modeling of earthquake ground motion with emphasis on computational efficiency of four basic types of the finite-difference schemes and their optimization. Special attention is paid to incorporation of the realistic attenuation that has not been affordable so far in the 3-D finite-difference modeling, i.e., which could not be included without significant memory and time optimization.

We do not discuss here such problems of the finite-difference schemes as consistency of the schemes on a free surface and internal material discontinuities, and matching between two grids, that affect accuracy of the schemes. Similarly we do not compare computational time requirements. Both problems require more space and special treatment.

2. Equations of Motion in Perfectly Elastic Medium

Consider a Cartesian coordinate system (x_1, x_2, x_3). Let $\rho(\mathbf{x})$ be density, $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ Lamè elastic coefficients, $\mathbf{u}(\mathbf{x}, t)$ displacement vector, t time, $\tau_{ij}(\mathbf{x}, t)$; $i, j \in \{1, 2, 3\}$ stress-tensor and $\mathbf{f}(\mathbf{x}, t)$ body force per unit volume. Then the equation of motion reads

$$\rho u_{i,tt} = \tau_{ij,j} + f_i \,, \tag{2.1}$$

where $i, j \in \{1, 2, 3\}$, $u_{i,tt} = \partial^2 u_i / \partial t^2$, $\tau_{ij,j} = \partial \tau_{ij} / \partial x_j$, and so on. The summation convention for repeated subscripts is assumed. The stress tensor τ_{ij} in an isotropic medium is given by Hooke's law

$$\tau_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}) \,, \tag{2.2}$$

where $i, j, k \in \{1, 2, 3\}$. For the purpose of classification of the finite-difference schemes (that will be considered later) we can call the Eqs. (2.1) and (2.2) the *displacement-stress* formulation of the equation of motion.

Let $\dot{u}_i(\mathbf{x}, t)$ be the i-th particle-velocity component. Then we can define the *displacement-velocity-stress* formulation of the equation of motion

$$\rho \dot{u}_{i,t} = \tau_{ij,j} + f_i \,, \tag{2.3}$$

$$\dot{u}_i = u_{i,t} \,, \tag{2.4}$$

$$\tau_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}) \,. \tag{2.5}$$

If we do not calculate displacement using Eq. (2.4) we have to differentiate Hooke's law (2.5) with respect to time. We obtain the *velocity-stress* formulation

$$\rho \dot{u}_{i,t} = \tau_{ij,j} + f_i \,, \tag{2.6}$$

$$\tau_{ij,t} = \lambda \dot{u}_{k,k} \delta_{ij} + \mu(\dot{u}_{i,j} + \dot{u}_{j,i}) \,. \tag{2.7}$$

Finally, inserting Hooke's law into Eq. (2.1) we obtain the *displacement* formulation of the equation of motion

$$\rho u_{i,tt} = (\lambda u_{k,k})_i + (\mu u_{i,j})_j + (\mu u_{j,i})_j + f_i \,. \tag{2.8}$$

Here, $(\mu u_{j,i})_j = \partial(\mu \frac{\partial u_j}{\partial x_i}) / \partial x_j$, and so on. Later, x, y, z will be used instead of x_1, x_2, x_3. Similarly, u, v, w will be used instead of u_1, u_2, u_3.

3. Finite-Difference Schemes

The equation of motion can be solved using the finite-difference (FD) method. The FD method is a direct numerical method for solving differential equations. The application of the FD method includes construction of a discrete FD model of the problem, analysis of the FD model, and numerical computations.

The above-described formulations of equation of motion (2.1)–(2.8) can be used for constructing FD schemes. Correspondingly, we can call the FD schemes displacement-stress (DS), displacement-velocity-stress (DVS), velocity-stress (VS), and displacement (D) FD schemes, respectively. While displacement schemes on conventional grids have been used since the end of the sixties, e.g., see Ref. 10, the displacement scheme with a good level of consistency at material discontinuities has been developed only recently by Zahradník.[11] Madariaga[12] suggested a velocity-stress scheme on a staggered grid to study the seismic source and Virieux[13,14] adopted it for modeling the SH and P-SV elastic waves. Luo and Schuster[15] suggested a parsimonious staggered-grid scheme based on the displacement-stress formulation.

Displacement formulation usually is applied on a conventional grid (all displacement components and material parameters are defined at each grid point). The other formulations advantageously are applied on a staggered grid (different displacement/particle-velocity components, stress-tensor components and material parameters are defined in different grid positions).

Since full presentation of the FD schemes in the three-dimensional (3-D) case would be too lengthy and unnecessary, we will give here only simplified schemes. Let U be displacement, $\dot{U}$ particle velocity, $U_x(\dot{U}_x)$ spatial derivative of $U(\dot{U})$, T stress, M elastic modulus, and m time-level index. Then we can write the following simplified FD schemes:

displacement-stress

$$\begin{aligned} T^m &= M \cdot U_x^m \,, \\ U^{m+1} &= 2U^m - U^{m-1} + \frac{\triangle^2 t}{\rho} T_x^m \,, \end{aligned} \tag{3.1}$$

displacement-velocity-stress

$$T^m = M \cdot U_x^m \,, \tag{3.2}$$

$$\dot{U}^{m+1/2} = \dot{U}^{m-1/2} + \frac{\triangle t}{\rho} T_x^m \,, \tag{3.3}$$

$$U^{m+1} = U^m + \triangle t \dot{U}^{m+1/2} \,, \tag{3.4}$$

velocity-stress

$$\begin{aligned} T^m &= T^{m-1} + \triangle t M \dot{U}_x^{m-1/2} \,, \\ \dot{U}^{m+1/2} &= \dot{U}^{m-1/2} + \frac{\triangle t}{\rho} T_x^m \,, \end{aligned} \tag{3.5}$$

displacement

$$U^{m+1} = 2U^m - U^{m-1} + \frac{\triangle^2 t}{\rho} (M U_x^m)_x \,, \tag{3.6}$$

where $\triangle t$ is a time-step. All the schemes are second-order accurate in time while the order of approximation in space is not specified since the spatial-derivative terms are included only symbolically. In other words, the schemes are valid for the second-, fourth- or higher-order of approximation.

Detailed presentations of the schemes in the 3-D case can be found, e.g., in papers by Ohminato and Chouet[16] (second-order displacement-stress scheme), Graves[17] (fourth-order velocity-stress scheme), and Moczo *et al.*[18] (second-order displacement scheme). We do not know any paper presenting the displacement-velocity-stress scheme. In fact, we want to promote it by this article.

Regarding accuracy, we can divide the four schemes into a group consisting of the displacement-stress, displacement-velocity-stress and velocity-stress schemes, and the displacement scheme. While the displacement scheme has accuracy problems in media with high Poisson's ratio and large velocity contrasts, the three other schemes can be applied to such media.

On the other hand, Moczo *et al.*[18] demonstrated remarkable accuracy of their displacement scheme in media with $\alpha/\beta < 2$ and velocity contrast as large as 5. They showed that the scheme was capable to account for the position of an internal discontinuity more accurately than the three staggered-grid schemes. The level of accuracy in simulating the traction-free condition on a flat free surface was also shown to be very good. These properties of the particular displacement scheme are reasons why we include it in this review.

Let us note again that, generally, a displacement scheme can mean, in fact, a variety of different displacement schemes that differ in accuracy. The difference comes from different approximations of the mixed second spatial derivatives.

4. Memory Requirements in Perfectly Elastic Media

4.1. *Time integration*

It is clear from Eqs. (2.1)–(2.8) and FD schemes (3.1)–(3.6) that they differ in what is integrated in time. In the case of the displacement-stress and displacement schemes, time marching is applied to displacement. Displacements at two successive time-levels, say m and $m-1$, have to be stored in memory in order to update displacement at time-level $m+1$. In the case of the displacement-velocity-stress scheme time marching is applied to both displacement and particle velocity. Since, however, displacement and particle velocity are related by Eq. (3.4) and shifted in time by half-value of the time-step, only displacement at one time-level, m, and particle velocity at one time-level, $m-1/2$, have to be stored in memory in order to update displacement at time-level $m+1$ and particle velocity at time-level $m+1/2$. In the case of the velocity-stress scheme time marching is applied to both the particle velocity and stress. Similarly, as in the previous case, only particle velocity at one time-level, $m-1/2$, and stress at one time-level, $m-1$, have to be stored in order to update particle velocity at time-level $m+1/2$ and stress at time-level m. An important difference between the displacement-stress/displacement-velocity-stress and velocity-stress schemes comes from the simple fact that the stress tensor has six independent components while displacement and particle velocity only three.

4.2. *Point-to-point heterogeneity of the medium*

Assume first such heterogeneity of the medium that ρ, λ and μ can change between each two grid points. Then it follows from the derivation of the displacement-stress, displacement-velocity-stress and velocity-stress schemes that each grid position of the displacement/particle-velocity components should be assigned its value of density, say, $U/\dot{U}$, $V/\dot{V}$ and $W/\dot{W}$ grid positions should be assigned ρ^U, ρ^V and ρ^W density values, respectively. Similarly, we should assign three different values of shear modulus, μ^{xy}, μ^{xz} and μ^{yz} to three different grid positions of the stress-tensor components T_{xy}, T_{xz} and T_{yz}, respectively. Finally, two values, λ and μ, are assigned to the joint grid position of three diagonal stress-tensor components T_{xx}, T_{yy} and T_{zz}.

In the case of the displacement scheme, each grid point of a conventional grid is assigned one value of density and three values of $\lambda : \lambda^x, \lambda^y, \lambda^z$ and $\mu : \mu^x, \mu^y, \mu^z$. Here, λ^x, e.g., denotes a harmonic average of λ between two neighboring grid points in the x-direction.

4.3. *Homogeneous material cells*

Both the displacement-stress and velocity-stress schemes are used by all (as far as we know) with the assumption of a homogeneous medium within one grid cell. This means that only three material parameters, ρ, λ and μ, are assigned to each grid cell. In the case of the displacement scheme, keeping six effective parameters ($\lambda^x, \lambda^y, \lambda^z$ and μ^x, μ^y, μ^z) is essential for accuracy. Simplification would decrease not only the number of parameters but mainly the level of consistency at material discontinuities and consequently the overall accuracy.[18,19]

4.4. *Memory requirements*

We can now summarize memory requirements of the four FD schemes. Let MX, MY and MZ be the numbers of grid points/cells in the x-, y- and z- directions, respectively. Let p denote the number of bytes for the used real-value precision; $p = 4$ in single precision and $p = 8$ in double precision. Displacement, particle-velocity, stress-tensor components, and material parameters that have to be stored, as well as the numbers of bytes occupied by these quantities in the four types of the FD schemes are as follows:

Point-to-point heterogeneous medium

displacement-stress scheme

$$U^m, V^m, W^m, U^{m-1}, V^{m-1}, W^{m-1}\,,$$
$$\rho^U, \rho^V, \rho^W, \lambda, \mu, \mu^{xy}, \mu^{xz}, \mu^{yz}\,,$$
$$N^P_{DS} = p \cdot MX \cdot MY \cdot MZ \cdot 14\,, \tag{4.1}$$

displacement-velocity-stress scheme

$$U^m, V^m, W^m, \dot{U}^{m-\frac{1}{2}}, \dot{V}^{m-\frac{1}{2}}, \dot{W}^{m-\frac{1}{2}}\,,$$
$$\rho^U, \rho^V, \rho^W, \lambda, \mu, \mu^{xy}, \mu^{xz}, \mu^{yz}\,,$$
$$N^P_{DVS} = p \cdot MX \cdot MY \cdot MZ \cdot 14\,, \tag{4.2}$$

velocity-stress scheme

$$\begin{gathered}\dot{U}^{m-\frac{1}{2}}, \dot{V}^{m-\frac{1}{2}}, \dot{W}^{m-\frac{1}{2}}, T_{xx}^{m-1}, T_{yy}^{m-1}, T_{zz}^{m-1}, T_{xy}^{m-1}, T_{xz}^{m-1}, T_{yz}^{m-1}\,, \\ \rho^U, \rho^V, \rho^W, \lambda, \mu, \mu^{xy}, \mu^{xz}, \mu^{yz}\,, \\ N_{VS}^P = p \cdot MX \cdot MY \cdot MZ \cdot 17\,, \end{gathered} \tag{4.3}$$

displacement scheme

$$\begin{gathered} U^m, V^m, W^m, U^{m-1}, V^{m-1}, W^{m-1}\,, \\ \rho, \lambda^x, \lambda^y, \lambda^z, \mu^x, \mu^y, \mu^z\,, \\ N_D^P = p \cdot MX \cdot MY \cdot MZ \cdot 13\,, \end{gathered} \tag{4.4}$$

Heterogeneous medium consisting of homogeneous material cells

Since $\rho = \rho^U = \rho^V = \rho^W$ and $\mu = \mu^{xy} = \mu^{xz} = \mu^{yz}$ in this case, we have

$$N_{DS} = p \cdot MX \cdot MY \cdot MZ \cdot 9\,, \tag{4.5}$$

$$N_{DVS} = p \cdot MX \cdot MY \cdot MZ \cdot 9\,, \tag{4.6}$$

$$N_{VS} = p \cdot MX \cdot MY \cdot MZ \cdot 12\,, \tag{4.7}$$

instead of Eqs. (4.1)–(4.3).

It is very clear that the assumption of homogeneous material cells considerably reduces memory requirements. Given such parameterization of the medium, i.e., having only three values, ρ, λ, μ, assigned to one grid cell, Graves[17] suggested how to compute effective parameters $\rho^U, \rho^V, \rho^W, \lambda, \mu, \mu^{xy}, \mu^{xz}$, and μ^{yz} without additional memory requirements. Using numerical tests Graves[17] verified that application of effective parameters better accounts for material heterogeneity. For another example of treating homogeneous material cells see paper by Ohminato and Chouet.[16]

Compared to the velocity-stress scheme, the displacement-stress and displacement-velocity-stress schemes need only 82% of memory in the case of the point-to-point heterogeneity and only 75% in the case of the medium consisting of homogeneous cells. This, obviously, is significant.

The displacement-stress and displacement-velocity-stress schemes need the same memory. The use of the displacement-velocity-stress scheme has two advantages compared to the displacement-stress scheme: (1) we have both the displacement and particle velocity at the same grid positions at each time-level, (2) programming and code optimization is easier.

5. Additional Memory Requirements Due to Attenuation

Incorporation of the realistic models of attenuation in time-domain computations has been made possible thanks to methods developed by Day and Minster,[20] Emmerich and Korn[21] and Carcione *et al.*[22] Emmerich and Korn's approach is based on rheology of the generalized Maxwell body while Carcione[22] made use of the generalized Zener body. Both realistic attenuation models allow accounting for an arbitrary $Q(\omega)$ law and spatially varying attenuation.

Although possible in principle, incorporation of the realistic attenuation in the 3-D modeling requires considerable additional computer memory and time. Increase of memory requirements poses a serious problem. Therefore, Graves[17] suggested an approximate technique to include attenuation at minimal cost. We will briefly review both approaches and corresponding additional memory requirements.

5.1. *Approximate attenuation correction*

In the approximate technique suggested by Graves[17] the updated stress and particle velocity are multiplied at each time-step by the attenuation function $A = \exp(-\pi f_0 \frac{\triangle t}{Q_\beta})$ where Q_β is a spatially varying quality factor for the S waves at the reference frequency f_0 and $\triangle t$ is a time-step. Such attenuation function is correct for a plane wave in a homogeneous medium. Since Q_β is a linear function of frequency, the technique can give good results if the frequency range of the simulation is relatively narrow and centered around the frequency and if Q_β does not significantly differ from Q_α. On the other hand, an attractive feature of the technique is the minimum additional memory requirement that is, in bytes, $N_{VS}^{AA} = p \cdot MX \cdot MY \cdot MZ$, assuming, generally, different Q_β for different grid cells. The technique is also applicable to other three schemes with the same additional memory $N_{VS}^{AA} = N_{DS}^{AA} = N_{DVS}^{AA} = N_{D}^{AA}$. The attenuation function is applied to those field quantities that are integrated in time.

5.2. *Attenuation based on rheology of the generalized Maxwell body*

Incorporation of the attenuation based on rheology of the generalized Maxwell body means that the Hooke's law (2.2) is modified:

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}) - \sum_{l=1}^{n} \zeta_l^{ij} \,.$$

The anelastic functions ζ_l^{ij} are determined by equations

$$\dot{\zeta}_l^{ij} + \omega_l \zeta_l^{ij} = \omega_l[\lambda Y_l^\lambda u_{k,k}\delta_{ij} + \mu Y_l^\mu (u_{i,j} + u_{j,i})] \,; \quad l = 1, \dots, n \,,$$

where $\omega_l, l = 1, \dots, n$, are the angular relaxation frequencies. The coefficients Y_l^λ and Y_l^μ, $l = 1, \dots, n$, are obtained from the systems of equations

$$\sum_{l=1}^{n} \frac{\omega_l \tilde{\omega}_k + \omega_l^2 \tilde{Q}_\eta^{-1}(\tilde{\omega}_k)}{\tilde{\omega}_k^2 + \omega_l^2} Y_l^\eta = \tilde{Q}_\eta^{-1}(\tilde{\omega}_k) \,, \quad k = 1, \dots, 2n-1 \,,$$

$$\eta \in \{\alpha, \beta\} \,, \quad Y_l^\mu = Y_l^\beta \,, \quad Y_l^\lambda = \frac{\alpha^2 Y_l^\alpha - 2\beta^2 Y_l^\mu}{\alpha^2 - 2\beta^2} \,,$$

where $\tilde{Q}_\alpha(\tilde{\omega}_k)$ and $\tilde{Q}_\beta(\tilde{\omega}_k)$, $k = 1, \ldots, 2n-1$, are desired values of the quality factors for the P and S waves, respectively, at the specified frequencies $\tilde{\omega}_k$. (More detailed presentation in the case of the displacement FD scheme can be found in paper by Moczo *et al.*[9])

In the case of the most efficient FD schemes, i.e., in the case of the displacement-stress and displacement-velocity-stress schemes, the simplified FD schemes (see Eqs. (3.1) and (3.2)) are modified as follows:

$$\zeta_l^{m+\frac{1}{2}} = \frac{2-\omega_l \triangle t}{2+\omega_l \triangle t}\zeta_l^{m-\frac{1}{2}} + \frac{2\omega_l \triangle t}{2+\omega_l \triangle t} Y_l^M U_x^m\,, \quad l = 1, \ldots, n\,,$$

$$\zeta_l^m = \frac{1}{2}(\zeta_l^{m-\frac{1}{2}} + \zeta_l^{m+\frac{1}{2}})\,,$$

$$T^m = MU_x^m - \sum_{l=1}^{n} \zeta_l^m\,,$$

and

$$U^{m+1} = 2U^m - U^{m-1} + \frac{\triangle^2 t}{\rho} T_x^m$$

in the displacement-stress scheme or

$$\dot{U}^{m+\frac{1}{2}} = \dot{U}^{m-\frac{1}{2}} + \tfrac{\triangle t}{\rho} T_x^m\,,$$

$$U^{m+1} = U^m + \triangle t \dot{U}^{m+\frac{1}{2}}$$

in the displacement-velocity-stress scheme. Here, ζ_l^m represents an appropriate anelastic function $\zeta_l^{ij}, i, j \in \{1,2,3\}$ and $l = 1, \ldots, n$, similarly as, e.g., M stands for an appropriate elastic modulus.

Incorporation of the above-described attenuation in the displacement-stress and displacement-velocity-stress schemes requires storing of the following additional quantities for each grid cell:

$$Y_l^\lambda, Y_l^\mu, \zeta_l^{xx}, \zeta_l^{yy}, \zeta_l^{zz}, \zeta_l^{xy}, \zeta_l^{xz}, \zeta_l^{yz}\,, \quad l = 1, \ldots, n\,.$$

The corresponding number of bytes is

$$N_{DS}^A = N_{DVS}^A = p \cdot MX \cdot MY \cdot MZ \cdot 8n\,.$$

The velocity-stress scheme requires the same additional memory, i.e.,

$$N_{VS}^A = N_{DS}^A = N_{DVS}^A\,.$$

The only slight difference is in the definition of anelastic functions due to time derivative of the stress tensor in the velocity-stress formulation.

For the sake of completeness, let us also mention the case of the displacement scheme. Additional quantities to be stored for each grid point are (see Ref. 9)

$$Y_l^\lambda, Y_l^\mu, \zeta_l^U, \zeta_l^V, \zeta_l^W\,, \quad l = 1, \ldots, n\,,$$

and the corresponding number of bytes is

$$N_D^A = p \cdot MX \cdot MY \cdot MZ \cdot 5n\,.$$

Here we assume that the attenuating properties of the medium are sufficiently described by some effective point values Y_l^λ and $Y_l^\mu, l = 1, \ldots, n$ assigned to each grid point. Let us recall that elastic properties in our displacement scheme are described using harmonic averages evaluated along the grid legs between two neighboring grid points.

Emmerich and Korn[21] demonstrated that for many applications it is sufficient to consider three relaxation frequencies, i.e., $n = 3$. Then the additional memory requirements are:

displacement-stress, displacement-velocity-stress, velocity-stress

$$N_{DS}^A = N_{DVS}^A = N_{VS}^A = p \cdot MX \cdot MY \cdot MZ \cdot 24\,,$$

displacement

$$N_D^A = p \cdot MX \cdot MY \cdot MZ \cdot 15\,.$$

Obviously, the additional memory is too large. Therefore, Zeng[23] and Day[24] developed a new method that allows to incorporate the realistic attenuation with significantly lower memory requirements. In Day's[24] approach, the anelastic functions ζ_l^{ij} are spatially distributed in a coarse staggered manner. One anelastic function ζ_l^{ij} for one relaxation frequency ω_l is distributed with a spatial period of $2h$, h being a grid spacing. Consequently, n, the number of relaxation frequencies, is 8. Consider, e.g., a grid cube $h \times h \times h$ with the stress-tensor component T^{xz} located at the cube's 8 corners. Only one of the 8 $\zeta_l^{ij}, l = 1, \ldots, 8$, anelastic functions is assigned to each of the 8 positions (cube's corners), say, ζ_1^{xz} is assigned to one position, ζ_2^{xz} to other position, etc. Consequently, the total number of $\zeta_l^{xz}, l = 1, \ldots, 8$, in the whole grid is $8\ \frac{MX}{2}\frac{MY}{2}\frac{MZ}{2} = MX \cdot MY \cdot MZ$. Since we have six independent stress-tensor components, the total number of the anelastic functions $\zeta_l^{ij}, l = 1, \ldots, 8$, in the whole grid is $6 \cdot MX \cdot MY \cdot MZ$. Since Y_l^λ and Y_l^μ are distributed in the same manner (Y_l^λ and Y_l^μ are assigned to the set of $\zeta_l^{ij}, i, j, \in \{1, 2, 3\}$, for one relaxation frequency ω_l), the total number of Y_l^λ and Y_l^μ, $l = 1, \ldots, 8$, in the whole grid is $2 \cdot MX \cdot MY \cdot MZ$. The additional memory due to attenuation in the Day's[24] approach in the displacement-stress, displacement-velocity-stress and velocity-stress schemes is

$$N_{DS}^{AD} = N_{DVS}^{AD} = N_{VS}^{AD} = p \cdot MX \cdot MY \cdot MZ \cdot 8\,.$$

The additional memory in the case of the displacement scheme is

$$N_D^{AD} = p \cdot MX \cdot MY \cdot MZ \cdot 5\,.$$

We see that the additional memory in the Day's[24] approach is equivalent to the case of just one relaxation frequency in the original Emmerich and Korn's[21] approach.

5.3. *Total memory requirements*

We can now summarize the total memory requirements for all four considered types of the FD schemes in the case of a heterogeneous medium consisting of homogeneous cells and attenuation based on rheology of the generalized Maxwell body in the Day's[23] approach:

displacement-stress, displacement-velocity-stress

$$N_{DS}^{T} = N_{DVS}^{T} = p \cdot MX \cdot MY \cdot MZ \cdot 17\,, \tag{5.1}$$

velocity stress

$$N_{VS}^{T} = p \cdot MX \cdot MY \cdot MZ \cdot 20\,, \tag{5.2}$$

In the case of the displacement scheme we consider point-to-point heterogeneity of the medium. The total required memory is

$$N_{D}^{T} = p \cdot MX \cdot MY \cdot MZ \cdot 18\,, \tag{5.3}$$

Comparing Eqs. (5.1)–(5.3) with Eqs. (4.5)–(4.7), it is obvious that the required total memory is too large in all cases. It is not difficult to estimate that the 3-D FD modeling of large-scale problems including realistic attenuation would be hardly affordable. An improvement can be achieved through memory optimization.

6. Optimization of the Finite-Difference Modeling

There is a variety of approaches to lower computational time and memory. Reduction of the total number of grid points and consequently reduction of core memory and computational time can be achieved by using fourth or higher-order scheme (e.g., Refs. 17, 25 and 26), grid with a varying size of grid spacings (e.g., Refs. 27 and 28) and combined/discontinuous grids (e.g., Refs. 9, 29–31). Reduction of core memory and use of disk memory is possible by applying core memory optimization.[17,32] Reduction of both core and disk memory and their balanced use is achieved by applying combined memory optimization, CDMO.[18,33] Parallel programming is now almost necessary to speed up computations for large models (e.g., Ref. 32).

Before we continue, let us briefly mention core and combined memory optimizations. Core memory optimization (as described by Graves[17] for the velocity-stress scheme) consists in keeping only a limited number of grid planes in core memory and performing a maximum possible number of time updates for these planes. The subset of planes repeatedly moves throughout the entire model space and particle velocity and stress are successively (plane by plane) and periodically overwritten in disk. (In the case of the displacement-stress scheme and attenuation displacement and anelastic functions would be periodically overwritten in disk.) Core memory is significantly reduced, however, disk memory requirements can become very large. Moreover, large number of the input/output operations increases computational time and creates a bottleneck of the computations.

Combined memory optimization, CDMO,[33,18] naturally comprises both core and disk memory optimizations and allows a balanced use of core and disk memory. In disk memory optimization, the wavelet transform is applied first to a 2-D array of the displacement (separately to each component) in order to decrease the information entropy. Second, data compression is performed in the wavelet domain. Then, e.g., only $2 \cdot 3$ streams of zeros and ones are stored and overwritten in disk instead of $2 \cdot 3 \cdot MX \cdot MY$ displacement values which have to be stored and overwritten in a pure core memory optimization for each grid plane and each component. (Here 2 means 2 time-levels, 3 means 3 displacement components, and we assume that a moving subset of planes is made of horizontal grid planes.) Thus, CDMO significantly reduces the total number of the input/output operations. Moreover, an increase of the CPU time due to one passage of the subset of planes with compression was in all numerical experiments always smaller than 0.75% of the time necessary for one passage without compression.[18,33] The time is, of course, computer- and code- dependent.

Let us now focus on the displacement-stress and displacement-velocity-stress schemes and show how different optimizations can make 3-D modeling in viscoelastic heterogeneous medium more efficient. The goal is to reduce memory requirement given by Eq. (5.1) as much as possible. We will do it in several steps.

6.1. *Material cell types*

Assuming a homogeneous medium within one grid cell we can consider the whole model to be composed of material cells of several types. Each grid cell is assigned a single integer number representing one of material cell types. The required total memory is then

$$N_{DS}^{T} = N_{DVS}^{T} = MX \cdot MY \cdot MZ \cdot (12p + q) + 5pK\,,$$

where 12 represents 3 displacement components at 2 time-levels (or 3 displacement and 3 particle-velocity components at 1 time-level) plus 6 anelastic functions $\zeta_l^{ij}, l = 1, \ldots, 8$, q is the number of bytes for the used integer value, 5 represents material parameters $\rho, \lambda, \mu, Y_l^{\lambda}$ and $Y_l^{\mu}, l = 1, \ldots, 8$, and K is the number of types of material cells. Obviously, in most cases $5pK$ is negligible and we will omit it.

6.2. *CDMO — combined memory optimization*

If we apply combined memory optimization we can distinguish the number of bytes in core memory, $COREM$, and the number of bytes in disk memory, $DISKM$. $COREM$ depends on the number of grid planes, say, NP, that are kept in core memory at one time. If we assume that the subset of NP planes is made of the horizontal grid planes,

$$COREM = MX \cdot MY \cdot NP \cdot (12p + q)\,. \tag{6.1}$$

$DISKM$ depends on the compression ratio CR that is determined by the wavelet compression. Since the wavelet compression is applied separately to each displacement/particle-velocity component and anelastic function at one plane at one time-level, CR represents a minimum of compression ratios for all components, anelastic functions, planes and time-levels. Disk memory is then

$$DISKM = MX \cdot MY \cdot MZ \cdot \left(\frac{12p}{CR} + q\right).$$

6.3. $h \times h \times h\_3h \times 3h \times 3h$ *discontinuous grid* + *CDMO*

Neither use of material cell types nor CDMO reduces the actual number of the grid cells. As mentioned earlier there are several ways how to do it. Here we consider a discontinuous grid whose upper part (covering region with lower velocities) is the $h \times h \times h$ grid and lower part (covering region with larger velocities) is the $3h \times 3h \times 3h$ grid. The size $3h$ of the grid spacing in the coarser grid is due to the structure of the scheme on a staggered grid. Let MZH be the number of grid cells in the z-direction in the upper $h \times h \times h$ grid. Let $MZ3H$ be the number of grid cells in the z-direction in the lower $3h \times 3h \times 3h$ grid. Then

$$COREM = MX \cdot MY \cdot NP \cdot (12p + q),$$

and

$$DISKM = \left[MX \cdot MY \cdot MZH + \left(\frac{MX-1}{3} + 1\right) \cdot \left(\frac{MY-1}{3} + 1\right) \cdot MZ3H\right] \left(\frac{12p}{CR} + q\right).$$

Here, $COREM$ is the same as that given by Eq. (6.1) for the uniform $h \times h \times h$ grid since the upper part of the discontinuous grid is the $h \times h \times h$ grid. Obviously, $COREM$ in the lower coarser grid is smaller.

7. Numerical Example

We can illustrate the above formulae for memory requirements on the example of modeling ground motion during the January 17, 1995 Hyogoken-Nanbu (Kobe) earthquake. We consider the same grid as was used for the elastic model in the simulation by Kristek *et al.*[34] The parameters of the model as well as memory requirements are given in Table 1. Since we use the fourth-order displacement-stress or displacement-velocity-stress FD scheme we take six grid spacings per minimum wavelength.[35]

Large memory requirements in the elastic modeling without optimization, given in Table 1, clearly illustrate necessity of memory optimization. This necessity is further stressed by significant increase of memory requirement due to inclusion of the realistic attenuation despite efficiency of the Day's[24] approach. Moreover, the grid model covers only part of the Osaka basin (see Ref. 34) and considered minimum S-wave velocity, $\beta_{\min}$, is, in fact, larger than the actual velocity in the uppermost layer. Artificial reduction of the computational region as well as the used minimum S-wave velocity obviously reduce numerical cost. This one more time underlines necessity of memory optimization.

Table 1. Memory requirements in the case of simple uniform grid and three levels of optimization in the fourth-order displacement-stress and displacement-velocity-stress finite-difference modeling. MX, MY, MZ — the numbers of grid cells in the x-, y- and z-directions, MZH — the number of grid cells in the z-direction in the upper $(h \times h \times h)$ part of the $h \times h \times h\_3h \times 3h \times 3h$ discontinuous grid, $MZ3H$ — the number of grid cells in the z- direction in the lower $(3h \times 3h \times 3h)$ part of the $h \times h \times h\_3h \times 3h \times 3h$ discontinuous grid, h- grid spacing, $\beta_{\min}$ — the minimum S-wave velocity, f_{ac} — the maximum frequency up to which the computation should be sufficiently accurate, NP — the number of horizontal grid planes in the subset of planes that is kept in core memory at one time, CR — minimum compression ratio, p — the number of bytes for the used real-value precision, q — the number of bytes for the used integer value, $COREM$ — core memory requirements, $DISKM$ — disk memory requirements, $CDMO$ — combined memory optimization, GMB — attenuation — attenuation based on rheology of the generalized Maxwell body.

$MX = 1054$, $MY = 247$, $MZ = 501$, $MZH = 47$, $MZ3H = 151$,
$h = 55$ m, $\beta_{\min} = 333$ m/s, $f_{ac} \leq 1Hz$,
$NP = 10$, $CR = 12$, $p = 4$, $q = 2$

	Elastic	GMB – Attenuation
uniform $h \times h \times h$ grid		
COREM	4478 MB[a]	8458 MB
DISKM	–	–
material cell types		
COREM	3234 MB	6219 MB
DISKM	–	–
CDMO		
COREM	65 MB	124 MB
DISKM	498 MB	746 MB
$h \times h \times h\_3h \times 3h \times 3h$ + CDMO		
COREM	65 MB	124 MB[b]
DISKM	64 MB	95 MB[b]

[a] $COREM$ in the case of the approximate attenuation correction would be 4975 MB.All other memory requirements would be the same as in the elastic case.
[b] Total memory requirements in the case of the $h \times h \times h\_3h \times 3h \times 3h$ grid without $CDMO$ would be 794 MB.

8. Conclusions

The 3-D finite-difference elastic and viscoelastic modeling of an earthquake ground motion in large-scale models of sedimentary basins requires considerable computer memory and time. The use of an efficient finite-difference scheme (e.g., the fourth-order displacement-stress scheme on an uniform staggered grid) is not sufficient, especially, if realistic attenuation is to be considered.

Fortunately, methods to reduce memory requirements and/or computational time in the finite-difference modeling have been developed recently.

We reviewed memory requirements of four basic types of the finite-difference schemes on the uniform grids in the cases of the 3-D perfectly elastic modeling, approximate attenuation correction and realistic attenuation.

We have presented application of the powerful optimization techniques to the 3-D fourth-order displacement-stress and displacement-velocity-stress modeling in the case of realistic attenuation. The techniques include description of a medium using material cell types, combined memory optimization, and combination of a discontinuous grid with combined memory optimization. We have shown that application of such powerful optimization significantly reduces memory requirements in simulations of an earthquake ground motion in realistic large-scale models. Implementation of the optimization techniques is not in contradiction with parallelization.

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